Lock-In Amplifier Theory

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Abstract

Lock-in detection is frequently used to increase signal-to-noise ratio of weak signals obscured by relatively strong background noise. Typically, one uses time-harmonic modulation of some experimental parameter and filters the observable quantity at the same quantity. Interestingly, though, filtering at higher harmonics of the modulation frequency yields additional information, whose exact nature depends on the particular setup.

Here I discuss under which conditions one expects to measure with higher harmonics the *original signal* or its *derivative* with respect to the modulating parameter.

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1.1 Overview

Fig. 1 depicts an experimental measurement employing a lock-in detection scheme. As an idealization, we assume the wave generator produces a perfect sine wave, usually a voltage signal of frequency f and

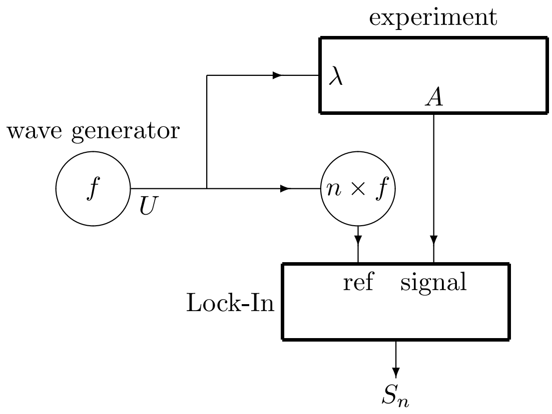

Figure 1: Principal experimental setup of lock-in detection. A wave generator is used to create a pure sine wave of frequency f to influence an experimental parameter λ, which in turn causes periodic changes in the observable A. In the lock-in amplifier, this observed signal is multiplied with a reference signal derived from the wave generator's wave as an integer harmonic nf to obtain a value for the signal S_n.

amplitude U_0 around an average value $\langle U \rangle$

$$U(t) \quad = \quad \langle U \rangle + U_0 \sin(2\pi f t) \ , \tag{1}$$

which is used in two ways.

Firstly, U *drives* some controllable experimental parameter λ with the exact same frequency f. Note that, despite its harmonic driver, this parameter need not necessarily perform a pure harmonic oscillation at f. It might contain also higher harmonics $n \times f$. Secondly, U (or possibly a generated higher harmonic of it) is used as a *reference* signal for the lock-in amplifier.

The observable A—viewed as a function of λ—is indirectly also a function of time,

$$A \quad = \quad A(\lambda) = A(\lambda(t)) \ . \tag{2}$$

Like λ, A will be periodic in time, with the same frequency f that U dictates—*with or without* non-zero contributions at higher harmonics.

1.2 Mathematical analysis of the lock-in operation

Mathematically, the lock-in amplifier may be described as a physical realization of a multiplication of the reference and the observed signal and a subsequent averaging, *i.e.*, integration of this product signal over relatively *long* time, *i.e.*over many periods of the fundamental frequency, $T = m/f$, where $m \gg 1$ is an integer;

$$S_n \quad = \quad \frac{1}{T}\int_0^T \sin(2\pi n f t + \delta\phi)\, A(\lambda(t)) dt \ . \tag{3}$$

Here, $\delta\phi$ is the phase shift between the reference and the observed signal, due to, for instance, different signal path lengths or delayed response in the action $\lambda \to A$.

In order to facilitate evaluation of this integral, and to understand the significance of the final signal S_n, we perform two steps to expand the experimental observable into relevant components:

- Expansion into a Taylor series of the experimental parameter λ. (This explains why sometimes observation at the higher harmonics is equivalent to observation of higher derivatives.)

- Expansion into a Fourier series in time t. (This explains why sometimes observation at any higher harmonics is equivalent to simple observation at the first harmonic.)

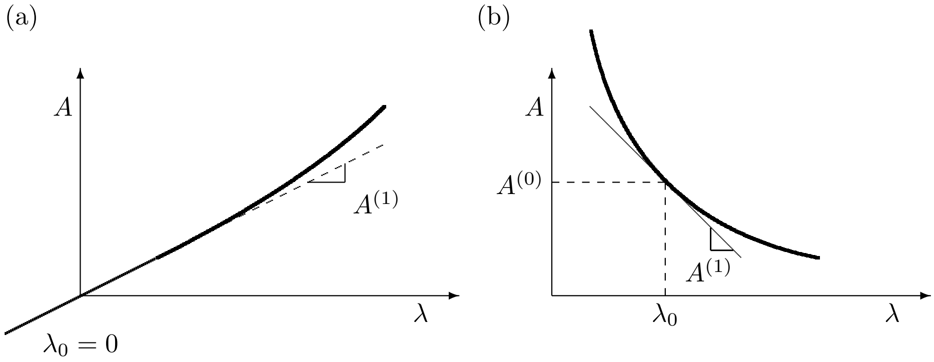

Figure 2: Typical dependences of the observable A on the physical parameter λ. The case (a), on the one hand, depicts the frequently encountered situation where an observable is strictly proportional over a large range of values of the parameter λ. This case describes, for instance, optical scattering experiments where the magnitude of the scattered light ($A = \mathbf{E}_{sca}$) depends linearly on the magnitude of the incident light ($\lambda = \mathbf{E}_{inc}$). Here we may assume only $A^{(1)}$ is non-zero. Case (b), on the other hand, is the more general case of a "smooth", though not flat response curve. Around the set point λ_0 the dependence is linear only for very small variations of λ. For larger variations, we may need to keep higher orders of the expansion $A^{(\nu>1)}$. An example of this situation is the dependence of a observable current ($A = I$) flowing through contacts attached to a sample that have been biased with a specific controllable voltage ($\lambda = V$).

1.2.1 Taylor Series of $A(\lambda)$

Expanding $A(\lambda)$ into a general Taylor series around $\lambda_0 = \lambda(t = 0)$, we obtain

$$A(\lambda) \quad = \quad \sum_{\nu=0}^{\infty} \frac{1}{\nu!} A^{(\nu)} \left(\lambda - \lambda_0\right)^{\nu} \ . \tag{4}$$

Here, $A^{(\nu)} = (\partial/\partial\lambda)^{\nu} A|_{\lambda=\lambda_0}$ stands as an abbreviation of the ν-th derivative of the observed quantity A with respect to the controlled quantity λ at the value λ_0. Often the dependence $A(\lambda)$ will be linear to a very good approximation over the extent that λ varies and we only need to care about the first derivative, because

$$\left|A^{(1)}\right| \quad \gg \quad \left|A^{(\nu)} \lambda_{\mu}^{\nu-1}\right| \ , \tag{5}$$

for any $\nu > 1$ and arbitrary μ.

1.2.2 Fourier Expansion of $\lambda(t)$

If we next expand the controllable parameter λ into its Fourier series,

$$\lambda(t) \quad = \quad \sum_{\mu=0}^{\infty} \lambda_{\mu} \sin(2\pi\mu f t + \delta\phi_{\mu}) \ , \tag{6}$$

we obtain, with the abbreviation $s_{\mu} \equiv \sin(2\pi\mu f t + \delta\phi_{\mu})$

$$A(t) \quad = \quad \sum_{\nu=0}^{\infty} \frac{1}{\nu!} A^{(\nu)} \left(\sum_{\mu=1}^{\infty} \lambda_{\mu} s_{\mu} \right)^{\nu} \ . \tag{7}$$

(Notice how λ_0 is eliminated by the interplay of Fourier and Taylor expansion so that the Fourier series starts with $\mu = 1$.)

Typically, the Fourier spectrum of the physical parameter will have only very few harmonics of non-negligible magnitude (see Fig. 3) and this allows us to truncate the expansion at a reasonable low harmonic. Rarely will we even need higher than second harmonics.

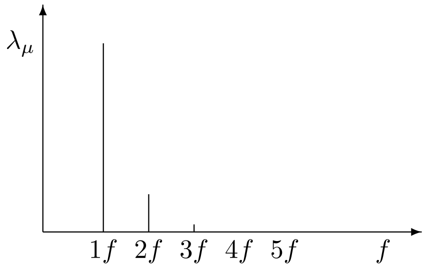

Figure 3: Typical amplitude spectrum of a controllable physical experimental parameter λ. Often, not even the lowest overtone frequency $(2f)$ is significant.

1.3 Discussion of $A(t)$

With increasing order, these various contributions to the observable A become rapidly intractably complicated. Explicitly, up to 3^{rd} order, these terms are

$$\begin{aligned} A(t) &= A^{(0)} && (8)\\ &+ A^{(1)}\left(\lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \ldots\right) && (9)\\ &+ \frac{1}{2}A^{(2)}\left(\lambda_1^2 s_1^2 + 2\lambda_1 s_1 \lambda_2 s_2 + \ldots\right) && (10)\\ &+ \frac{1}{6}A^{(3)}\left(\lambda_1^3 s_1^3 + \ldots\right) && (11)\\ &+ \ldots . && (12) \end{aligned}$$

Nevertheless, we can appreciate their basic behavior and how they contribute to S_n by expanding the products of sine functions into single sine functions using the algebraic identity

$$\sin(x)\sin(y) = \frac{1}{2}\left(\sin\left(x - y + \frac{\pi}{2}\right) + \sin\left(x + y - \frac{\pi}{2}\right)\right) . \quad (13)$$

In the present case, we find a sum of simple sine functions, each with a difference or a sum frequency

$$\begin{aligned} 2\sin(2\pi\mu_1 ft + \delta\phi_1)\sin(2\pi\mu_2 ft + \delta\phi_2) &= \sin\left(2\pi(\mu_1 - \mu_2)ft + (\delta\phi_1 - \delta\phi_2) + \frac{\pi}{2}\right) && (14)\\ &+ \sin\left(2\pi(\mu_1 + \mu_2)ft + (\delta\phi_1 + \delta\phi_2) - \frac{\pi}{2}\right) . && (15) \end{aligned}$$

Thus, we have

$$\begin{aligned} A(t) &= A^{(0)} && (16)\\ &+ A^{(1)}\left(\lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \ldots\right) && (17)\\ &+ \frac{1}{2}A^{(2)}\frac{1}{2}\left(\lambda_1^2 s_{1+1} + \lambda_1^2 s_{1-1} + 2\lambda_1\lambda_2 s_{1+2} + 2\lambda_1\lambda_2 s_{1-2} + \ldots\right) && (18)\\ &+ \frac{1}{6}A^{(3)}\frac{1}{4}\left(\lambda_1^3 s_{1+1+1} + \lambda_1^3 s_{1+1-1} + \lambda_1^3 s_{1-1+1} + \lambda_1^3 s_{1-1-1} + \ldots\right) && (19)\\ &+ \ldots , && (20) \end{aligned}$$

where, *e.g.*, s_{1+1-1} is an abbreviation of

$$\sin\left(2\pi(1 + 1 - 1)ft + (\delta\phi_1 + \delta\phi_1 - \delta\phi_1) - \frac{\pi}{2}(0 + 1 - 1)\right) . \quad (21)$$

To evaluate the final

$$S_n = \frac{1}{T}\int_0^T \sin\left(2\pi nft + \delta\phi\right) A(\lambda(t))dt \quad (22)$$

we first take notice of how the integrand expands into a sum of individual sine functions

$$\sin(2\pi nft + \delta\phi)\, A(\lambda(t)) = A^{(0)}(\tilde{s}_n) \tag{23}$$
$$+ A^{(1)}(\lambda_1 \tilde{s}_{n+1} + \lambda_1 \tilde{s}_{n-1} + \lambda_2 \tilde{s}_{n+2} + \lambda_2 \tilde{s}_{n-2} + \dots) \tag{24}$$
$$+ \frac{1}{2}A^{(2)}\frac{1}{2}\left(\lambda_1^2 \tilde{s}_{n+1+1} + \lambda_1^2 \tilde{s}_{n+1-1} + \lambda_1^2 \tilde{s}_{n-1+1} + \dots\right) \tag{25}$$
$$+ \frac{1}{6}A^{(3)}\frac{1}{4}\left(\lambda_1^3 \tilde{s}_{n+1+1+1} + \dots\right) \tag{26}$$
$$+ \dots , \tag{27}$$

where, we have abbreviated, *e.g.*,

$$\tilde{s}_{n-1+2} = \sin\left(2\pi(n-1+2)ft + (\delta\phi - \delta\phi_1 + \delta\phi_2) - \frac{\pi}{2}(0-1+1)\right) . \tag{28}$$

In general,

$$\sin(2\pi nft + \delta\phi)\, A(\lambda(t)) = \sum_{\{N\}} C_N \sin(2\pi Nft + \Delta\phi_N) , \tag{29}$$

where C_N is a constant factor of the form $A^{(\nu)}/\nu!2^{\mu}$, N is a particular, integer value for the sum of various harmonics $(n \pm \sum_j \pm\mu_j)$ and $\Delta\phi_N$ is the corresponding sum of phases $(\delta\phi \pm \sum_j \pm \left(\delta\phi_j - \frac{\pi}{2}\right))$.

Next we recall that almost all integrands yield vanishing integrals for sufficiently large T,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T C_N \sin(2\pi Nft + \Delta\phi_N)dt = \begin{cases} C_N \sin(\delta\phi_N) & \text{if } N = 0 \\ 0 & \text{otherwise.} \end{cases} . \tag{30}$$

Only those terms contribute, for which $N = 0$, that is,

$$n = \mp \sum_j \mp\mu_j . \tag{31}$$

This basic relation allows us to sift through all terms to identify those that actually do contribute non-zero values.

1.3.1 Case $n = 1$

The most basic operation of a lock-in amplifier is the case where the fundamental frequency $1f$ is fed into the reference input. We are now in a position to answer the question what will be observed as $S_{n=1}$ at the output.

Among the terms $\sum_{\nu=1}^{\infty} \frac{1}{\nu!} A^{(\nu)} \left(\sum_{\mu=1}^{\infty} \lambda_\mu s_\mu\right)^{\nu}$ that contribute to $S_{n=1}$, we recognize immediately the single term that involves the first derivative,

$$A^{(1)}\lambda_1 \tilde{s}_{n-1} , \tag{32}$$

which will contribute most dominantly. There are, however other terms, like

$$\frac{1}{2}A^{(2)}\frac{1}{2}(\lambda_1\lambda_2 \tilde{s}_{n-1-2} + \lambda_2\lambda_3 \tilde{s}_{n+2-3} + \dots) \tag{33}$$
$$+ \frac{1}{6}A^{(3)}\frac{1}{4}\left(\lambda_1^3 \tilde{s}_{n-1+1-1} + \lambda_1\lambda_2^2 \tilde{s}_{n-1+2-2} + \dots\right) \tag{34}$$
$$+ \frac{1}{24}A^{(4)}\frac{1}{8}\left(\lambda_1^3\lambda_2 \tilde{s}_{n-1+2-1-1} + \lambda_6\lambda_1\lambda_2^2 \tilde{s}_{n-6+1+2+2} + \dots\right) \tag{35}$$
$$+ \dots , \tag{36}$$

As was stipulated in subsection 1.2, we usually may ignore these higher order terms, as their amplitude is comparatively small.

1.3.2 Case $n = 2$

Another frequently encountered operation of a lock-in amplifier is the observation with the first harmonic $2f$ as reference input. Again, we recognize immediately the single term that involves the first derivative,

$$A^{(1)}\lambda_2\tilde{s}_{n-2}\ . \tag{37}$$

Among the terms involving the *second* derivative,

$$\frac{1}{2}A^{(2)}\frac{1}{2}\left(\lambda_1^2\tilde{s}_{n-1-1} + \lambda_2\lambda_4\tilde{s}_{n+2-4} + \lambda_3\lambda_5\tilde{s}_{n+3-5} + \ldots\right)\ , \tag{38}$$

we expect the $\tilde{s}_{n-1-1}$ term to be dominant, as all others involve amplitudes of higher harmonics ($\lambda_{\mu>1}$) as factors. Similarly, we may ignore all other terms involving even higher derivatives.

Now the interesting question arises, which of the two terms will actually dominate S_2. The first-derivative related

$$A^{(1)}\lambda_2\tilde{s}_{n-2}$$

or the term

$$\frac{1}{2}A^{(2)}\frac{1}{2}\lambda_1^2\tilde{s}_{n-1-1}$$

that involves the second derivative? It depends on which of the two inequalities

$$\left|A^{(1)}\right| \gg \left|A^{(2)}\lambda_1\right|$$

and

$$|\lambda_1| \gg |\lambda_2|$$

is more severe. In the most general situation, there is no unique answer. However, as we will discuss in the next two subsections, we can identify two typical scenarios in which we are able to answer this question.

1.4 Perfectly linear coupling of λ and A

If the relation of driving parameter and observable quantity is perfectly linear, as is explicitly the case, for instance, in linear optics (see case (a) in Fig. 2), we have

$$A(\lambda) \quad = \quad A^{(0)} + A^{(1)}\lambda \tag{39}$$

and all $A^{(\nu>1)} \equiv 0$. The only observed signal at the lock-in's output is simply

$$S_n = \sin\left(2\pi nft + \delta\phi\right) A(\lambda(t)) \tag{40}$$
$$= A^{(1)}\lambda_n\tilde{s}_{n-n} \tag{41}$$
$$= A^{(1)}\lambda_n\cos(\delta\phi - \delta\phi_n)\ . \tag{42}$$

That is, regardless of the harmonic frequency nf we supply as reference, we always observe merely the first derivative of the $A(\lambda)$ relation.

1.5 Perfectly linear coupling of U and λ

If the relation of reference wave voltage and the driving parameter is perfectly linear, as is trivially the case, for instance, if they are identical (see case (b) in Fig. 2), we have

$$\lambda(t) \quad = \quad \lambda_0 + \lambda_1\sin(2\pi ft) \tag{43}$$

and all $\lambda_{\mu>1} \equiv 0$. As the observable is comprised of terms involving only λ_1, we have

$$\begin{aligned} S_n &= \sin(2\pi n f t + \delta\phi)\, A(\lambda(t)) && (44)\\ &= A^{(0)}\tilde{s}_n && (45)\\ &+ A^{(1)}\lambda_1\,(\tilde{s}_{n+1} + \tilde{s}_{n-1}) && (46)\\ &+ \frac{1}{2}A^{(2)}\frac{\lambda_1^2}{2}\,(\tilde{s}_{n+1+1} + \tilde{s}_{n+1-1} + \tilde{s}_{n-1+1} + \tilde{s}_{n-1-1}) && (47)\\ &+ \frac{1}{6}A^{(3)}\frac{\lambda_1^3}{4}\,(\tilde{s}_{n+1+1+1} + \tilde{s}_{n+1+1-1} + \ldots) && (48)\\ &+ \ldots\,. && (49) \end{aligned}$$

Thus we have

$$S_1 \approx A^{(1)}\lambda_1\tilde{s}_{n-1} = A^{(1)}\lambda_1 \cos(\delta\phi - \delta\phi_1)\ , \quad (50)$$

$$S_2 \approx \frac{1}{2}A^{(2)}\frac{\lambda_1^2}{2}\tilde{s}_{n-1-1} = -\frac{A^{(2)}\lambda_1^2}{4}\sin(\delta\phi - 2\delta\phi_1)\ , \quad (51)$$

$$S_3 \approx \frac{1}{6}A^{(3)}\frac{\lambda_1^2}{4}\tilde{s}_{n-1-1-1} = -\frac{A^{(3)}\lambda_1^3}{24}\cos(\delta\phi - 3\delta\phi_1)\ , \quad (52)$$

and so on. That is, observation with a reference frequency of nf results mainly in measuring the n-th derivative of the $A(\lambda)$ relation.