Chapter 10

Radar Antenna

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Antenna Concept

- Transducer point of view:

  A transducer between an electrical signal in a system and a propagating wave.

- Mode conversion point of view:

  A transition device between a guided wave and a space wave.

- Energy conversion point of view:

  A converter between photons and currents.

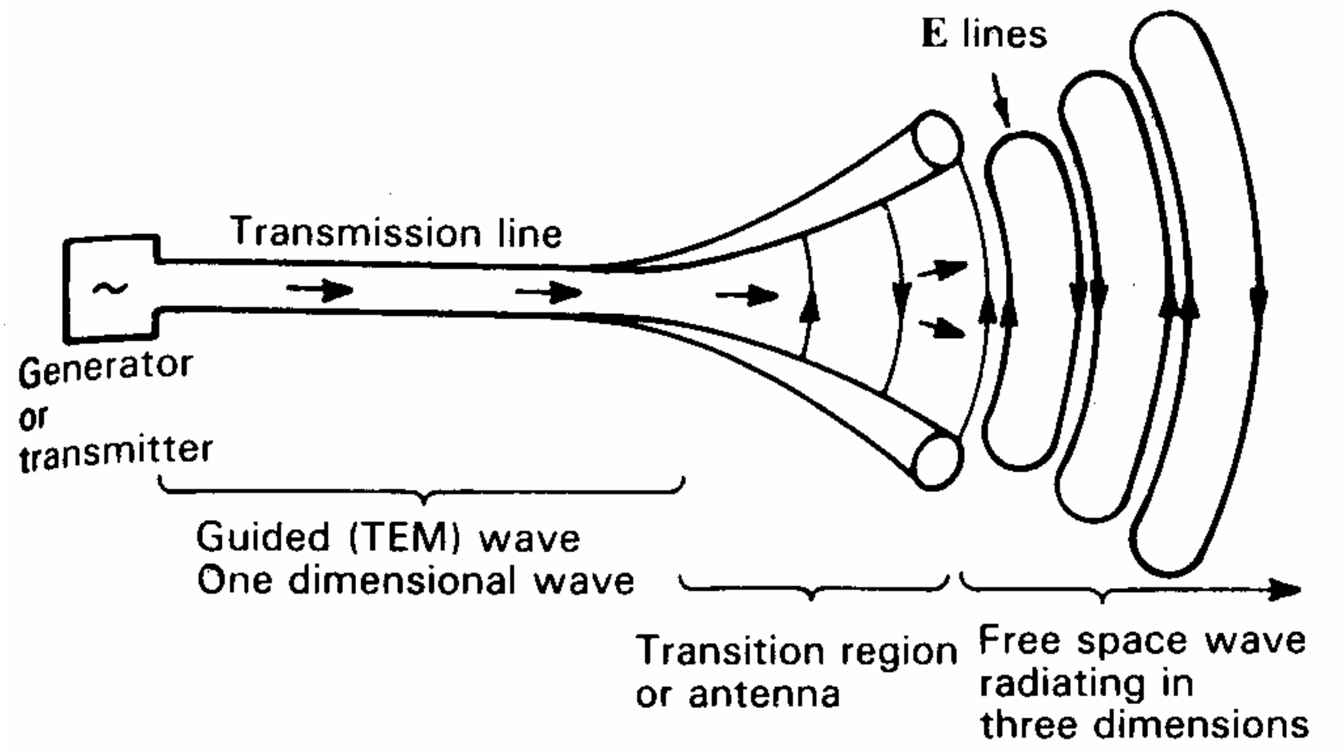
- Circuit point of view:

  One-port black box with input impedance.

- Signal transmission point of view:

  A spatial filter.

Conceptual antenna



Sources of Radiation

- Maxwell's equations in time-domain

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} : \text{Faraday's law}$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} : \text{Ampere's law}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t) : \text{Gauss' law}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0 : \text{Gauss' law}$$

$$\nabla \cdot \mathbf{J}(\mathbf{r},t) = -\frac{\partial \rho(\mathbf{r},t)}{\partial t} : \text{continuity equation}$$

- accelerated charges ⇔ time varying currents

  Electromagnetic Field Radiation

Maxwell's equations in Freq. Domain

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\, \mathbf{B}(\mathbf{r}) : \text{Faraday's law}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) + j\omega\, \mathbf{D}(\mathbf{r}) : \text{Ampere's law}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho\,(\mathbf{r}) : \text{Gauss' law}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 : \text{Gauss' law}$$

$$\nabla \cdot \mathbf{J}(\mathbf{r}) = -j\omega\, \rho\,(\mathbf{r}) : \text{continuity equation}$$

- # of unknowns : 16 (**E , D , B , H ,** $\mathbf{J_c}$, ρ)
- # of independent equations : 7
- # of necessary equations : 9 (constitutive relations)

$$\mathbf{D} = \varepsilon\, \mathbf{E}$$

$$\mathbf{B} = \mu\, \mathbf{H}$$

$$\mathbf{J_c} = \sigma\, \mathbf{E}$$

Vector and Scalar potentials

- Solution of Maxwell's equation in the free-space

$$\mathbf{E} = -j\varpi\,\mathbf{A} - \nabla\phi = -j\varpi\,\mathbf{A} + \frac{\nabla\nabla\cdot\mathbf{A}}{j\omega\mu_0\varepsilon_0}$$

$$\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A}$$

$$\text{where } \nabla^2\mathbf{A} + k_0^2\mathbf{A} = -\mu_0\mathbf{J} : \text{vector Helmholtz equation}$$

$$\nabla^2 V + k_0^2 V = -\frac{\rho}{\varepsilon_0} : \text{scalar Helmholtz equation}$$

- decoupling between components
- $\mathbf{J}_{x,y,z} \rightarrow \mathbf{A}_{x,y,z} \rightarrow \mathbf{H} \rightarrow \mathbf{E}$
- In cases of constant ε, μ, the Maxwell's equation is a linear system. The superposition rule can be applied to find field due to source distribution.

Solution of vector and scalar potentials

- Vector potential

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{[\mathbf{J}]}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'$$

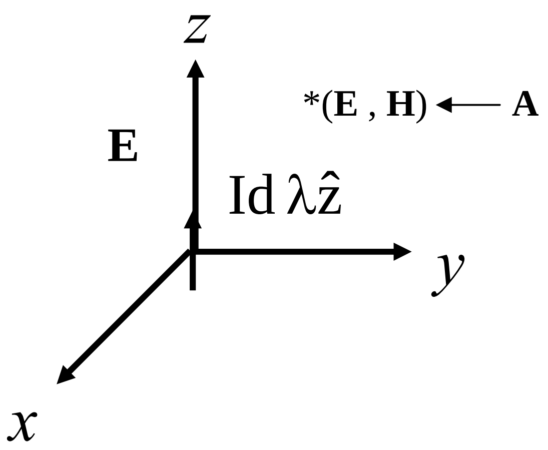
where $[\mathbf{J}]$ is retared current density.

- Scalar potential

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{[\rho]}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'$$

where $[\rho]$ is retared charge density.

Radiation from a short current filament



$$\mathbf{A}(r) = \frac{e^{-jk_0 r}}{4\pi r}\mu_0 I d\lambda\,\hat{z}$$

$$\mathbf{H} = \frac{Id\lambda \sin\theta}{4\pi}\left(\frac{jk_0}{r} + \frac{1}{r^2}\right)e^{-jk_0 r}\hat{\phi}$$

$$\mathbf{E} = \frac{jZ_0 I d\lambda}{2\pi k_0}\cos\theta\left(\frac{jk_0}{r^2} + \frac{1}{r^3}\right)e^{-jk_0 r}\hat{r}$$

$$-\frac{jZ_0 I d\lambda}{4\pi k_0}\sin\theta\left(-\frac{k_0^{\,2}}{r} + \frac{jk_0}{r^2} + \frac{1}{r^3}\right)e^{-jk_0 r}\hat{\theta}$$

Characteristics of short dipole(I)

- In far-field region $(r \gg \lambda_0, r \gg dl, r > \frac{2dl^2}{\lambda})$

$$\mathbf{E} = jZ_0 I d\lambda k_0 \sin\theta \frac{e^{-jk_0 r}}{4\pi r}\hat{\theta}$$

$$\mathbf{H} = jId\lambda k_0 \sin\theta \frac{e^{-jk_0 r}}{4\pi r}\hat{\phi}$$

1. proportional to I, dl(linear superposition)
2. propagate in the r-direction with E_θ, H_ϕ only(TEM wave)
3. $\frac{E_\theta}{H_\phi} = 377\,\Omega$ (E_θ, H_ϕ in time phase)
4. sin θ space-variation(signaling)
5. $\frac{1}{r}$ distance dependence(power conservation)
6. $e^{-jk_0 r}$ phase dependence(propagation delay)

Characteristics of short dipole(II)

- In near-field region(quasi-stationary)

$$\mathbf{E} = \frac{Qd\lambda}{4\pi}\left(\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right) \rightarrow \text{dipole fields}$$

$$\mathbf{H} = \frac{Id\lambda\sin\theta}{4\pi r^2}\hat{\phi} \rightarrow \text{static magnetic field}$$

1. Electric fields(E_r, E_θ) in time phase quadrature with H_ϕ: energy storage –> resonator
2. sin θ variation of E_θ, H_ϕ and cos θ variation of E_r components
3. $\frac{1}{r^2}$ or $\frac{1}{r^3}$ dependence: confinement of field in vicinity of dipole(radian distance $r \leq \frac{\lambda}{2\pi}$)

Radiation patterns (I)

- Representation of radiation as a function of direction in space
- Field patterns: need 3 patterns to represent the polarization

1. $E_\theta(\theta, \phi)$: intensity
2. $E_\phi(\theta, \phi)$: intensity
3. $\varphi_\theta(\theta, \phi)$ and $\varphi_\phi(\theta, \phi)$: polarization

- Normalized field patterns

$E_{\theta,\phi}(\theta, \phi)_n = E_{\theta,\phi}(\theta, \phi) / E_{\theta,\phi}(\theta, \phi)_{max}$

- Power pattern

$S(\theta, \phi) = \frac{1}{2} E(\theta, \phi) \times H(\theta, \phi)^*$: power per unit area

- Normalized power pattern

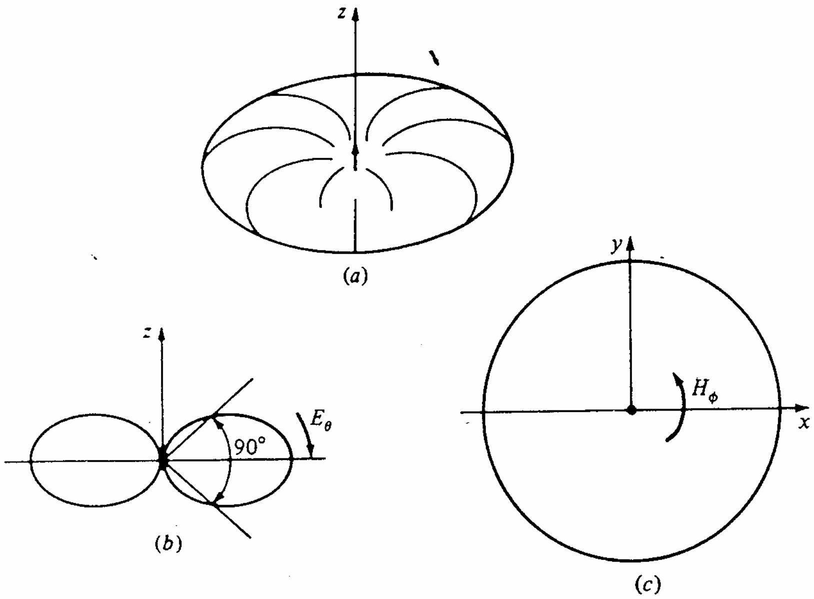
$P_n(\theta, \phi) = S(\theta, \phi) / S(\theta, \phi)_{max}$

Radiation patterns (II)

- 3-D pattern / 2-D pattern

1. E-plane pattern : plane including maximum radiation direction and electric field direction
2. H-plane pattern : plane including maximum radiation direction and magnetic field direction

short dipole



Radiation patterns (III)

- Plotting coordinates : - polar plot

  - rectangular plot

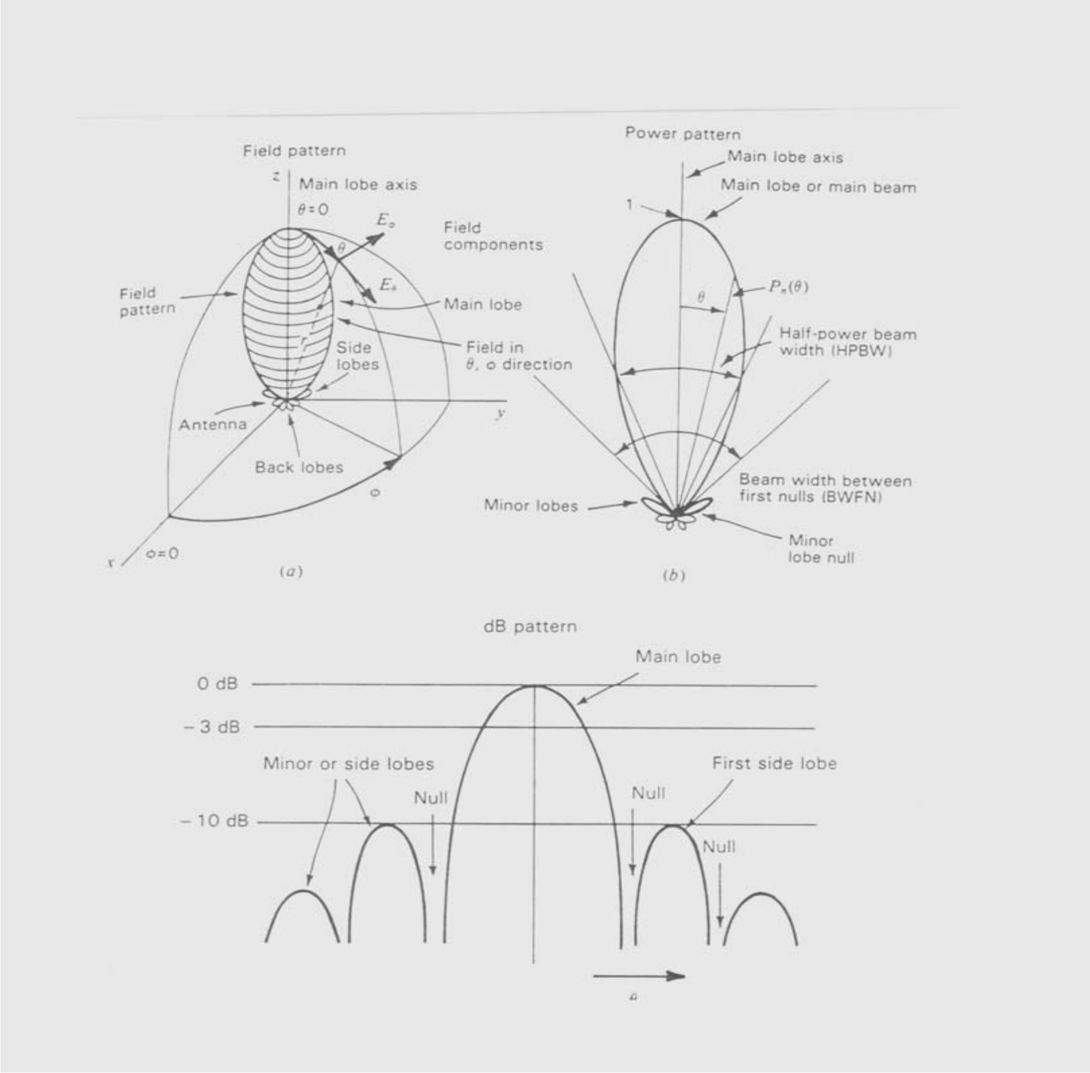
- Plotting scales: - linear scale

  - decibel scale : $dB = 10\log_{10} P_n(\theta,\phi)$

- Terminologies regarding radiation pattern

1. Main lobe(beam)
2. Minor or Side lobes(beams) : 1^{st}, 2^{nd} side lobes
3. Nulls
4. Half-power beam width(HPBW)
5. Beam width between first nulls(BWFN)

Pattern Examples



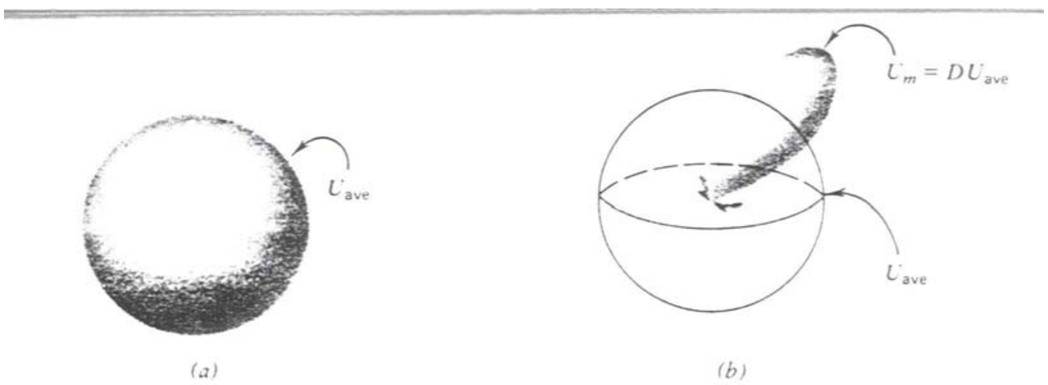
Radiation intensity, Directivity

- Radiation intensity : U (watt/steradian)

1. The power radiated from an antenna per unit solid angle.
2. $U(\theta,\phi)=S(\theta,\phi) \times r^2$: independent of the distance

- Directivity : D (dimensionless)

1. Ratio of the maximum radiation intensity to the average radiation intensity
2. $$D = \frac{U(\theta,\phi)_{\max}}{U_{av}} = \frac{U(\theta,\phi)_{\max}}{P_r / 4\pi}$$



Antenna beam solid angle

- Radiated power in terms of Ω_A:

$$p_r = \int S(\theta,\phi)\, r^2 \sin\theta \, d\theta \, d\phi = \int U(\theta,\phi) d\Omega = U_m \int \left|F(\theta,\phi)\right|^2 d\Omega$$

$$= U_m \Omega_A \approx U_m \beta_3 \phi_3$$

where β_3 and ϕ_3 are the antenna half-power (3-dB) beamwidths in either direction

- Directivity in terms of Ω_A

$$D = \frac{U(\theta,\phi)_{\max}}{U_{av}} = \frac{1}{\frac{1}{4\pi}\int \left|F(\theta,\phi)\right|^2 d\Omega}$$

$$= \frac{4\pi}{\Omega_A} \approx \frac{4\pi}{\beta_3 \phi_3}$$

- Short dipole : $\Omega_A = 2.67\pi$, D=1.5

Gain

- Gain : G
  1. Depends on both directivity and efficiency
  2. G=ηD

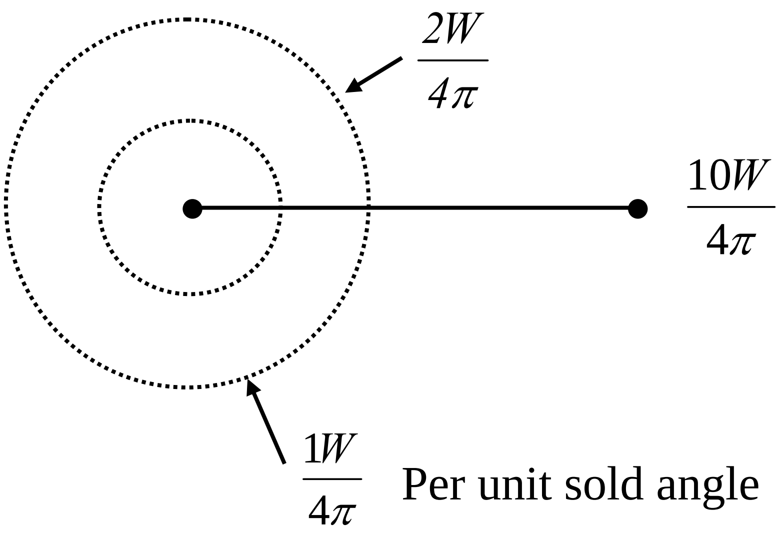
     where η =efficiency factor of antenna(0<η<1)
  3. Due to Ohmic losses in the antenna

  4. $$G = \frac{U_{\max}}{P_{in}/4\pi} = \frac{P_r}{P_{in}} \frac{U_{\max}}{P_r/4\pi} = \eta D, P_r = \eta P_{in}$$

EIRP

•Effective Isotropic Radiated Power(EIRP)

$$(\text{Gain} = 10\ ,\ P_{in} = 1\text{W}) = (\text{Gain} = 5\ ,\ P_{in} = 2\text{W})$$

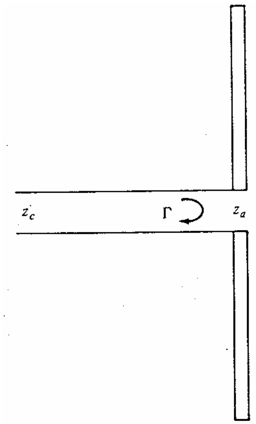


$$\text{EIRP} = \text{Gain} \times P_{in}$$

Per unit sold angle

Antenna Input Impedance

- Input impedance



$$\Gamma = \frac{Z_a - Z_c}{Z_a + Z_c}$$

$$Z_a = Z_c \frac{1+\Gamma}{1-\Gamma}$$

$$Z_a = \frac{V_a}{I_a}$$

$$= \frac{P_r + P_d + 2j\omega\left(W_m - W_e\right)}{\frac{1}{2}\left|I_0\right|^2}$$

Where

P_r : radiated power

P_d : ohmic loss of antenna

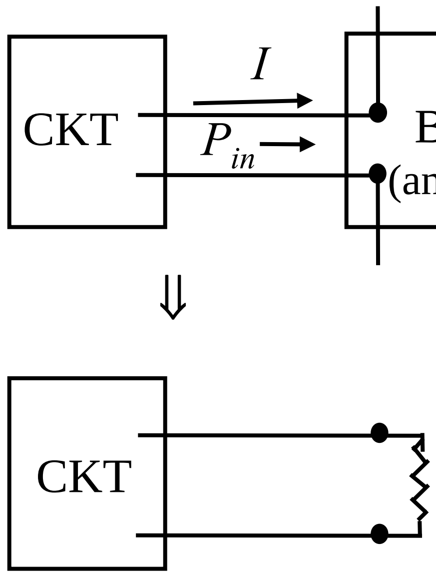
W_m : storage of magnetic energy

W_e : storage of electric energy

} Near field

Radiation resistance, Loss resistance

- Antenna input resistance :
  radiation loss and Ohmic loses



$$R_a = R_r + R_{loss}$$

$$P_{in} = P_r + P_{loss} = \frac{1}{2}|I|^2 R_a$$

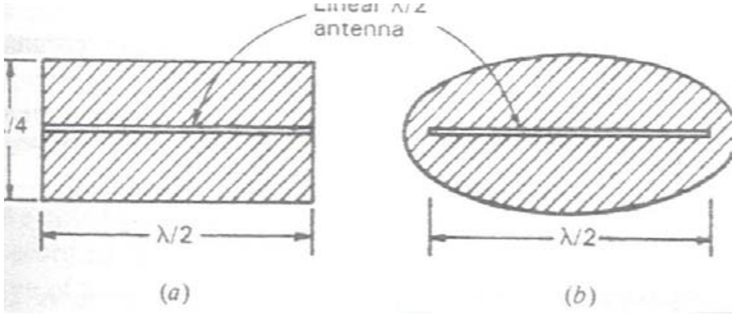
- Input reactance :
  near-field energy storage

$$jX = \frac{2j\omega(W_m - W_e)}{\frac{1}{2}|I_0|^2}$$

Effective aperture

- The ratio of the receiving power to the power density of the incident wave.

$$A_e = \frac{P_{rec}}{S}$$



- Aperture efficiency

$$\varepsilon_{ap} = \frac{A_e}{A_p}$$

- Gain vs effective aperture

$$G = \frac{4\pi}{\lambda^2} A_e$$

Number of Beam Positions

$$G = \eta D = \eta \frac{4\pi}{\beta_3 \phi_3} = \frac{4\pi}{\lambda^2} A_e$$

$$\therefore \beta_3 \phi_3 = \eta \frac{\lambda^2}{A_e} \approx \frac{\lambda^2}{A_e} \quad (\text{if } \eta \approx 1)$$

$$\text{Also, } \beta_3 \text{ and } \phi_3 \approx \frac{1}{\sqrt{A_e}}$$

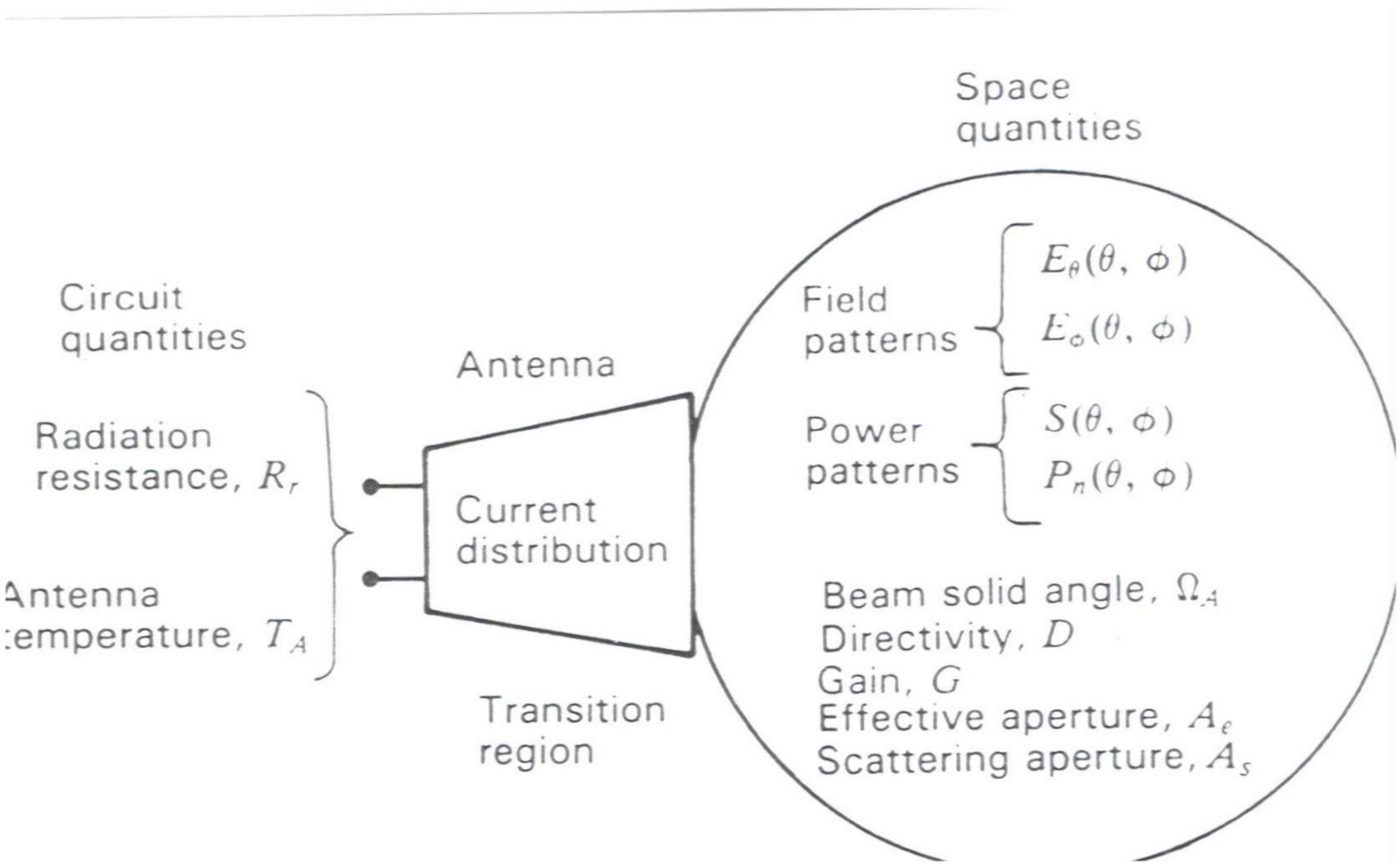
The number of beam positions to cover a volume V

$$N_{Beams} \ \phi \ \frac{V}{\beta_3 \phi_3}$$

When V represents the entire hemisphere,

$$N_{Beams} \ \phi \ \frac{2\pi}{\beta_3 \phi_3} \xrightarrow{\eta \approx 1} \frac{2\pi A_e}{\lambda^2} = \frac{G}{2}$$

Schematic of antenna parameters

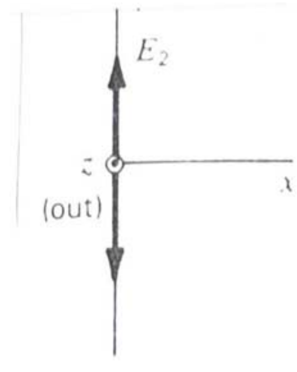


Polarization

- The orientation of the electric field E.

(1) linear polarization

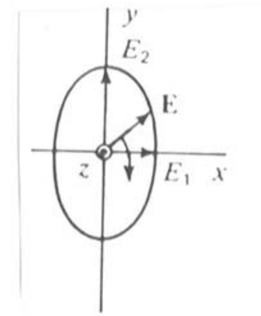
- horizontal
- vertical



(a)

(2) elliptical polarization

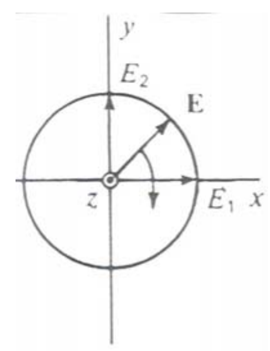
- axial ratio:



(b)

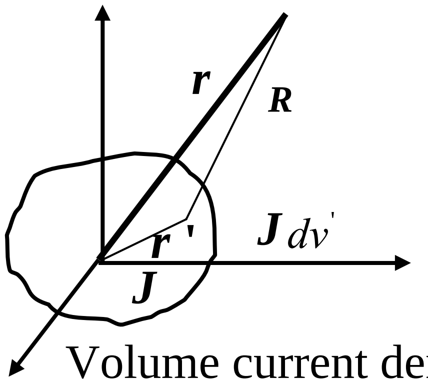
(3) circular polarization

- right-hand
- left-hand



(c)

Characteristics of arbitrary current distribution



Volume current density

$$\mathbf{A}(r) = \frac{\mu_0}{4\pi}\int_{v'} \frac{\mathbf{J}\, e^{-jk_0 R}}{R}\, dv'$$

far field approximation

$$R = |\mathbf{r} - \mathbf{r}'| \cong r - \underbrace{\hat{r}\cdot\mathbf{r}'}_{r'\cos\varphi}$$

$$\underline{\mathbf{E}(r) = \frac{jk_0 Z_0 e^{-jk_0 r}}{4\pi r}}\quad \underline{\mathbf{F}(\theta,\phi)} = \frac{jk_0 Z_0 e^{-jk_0 r}}{4\pi r}\int_{v'} \underline{\{\hat{r}\mathbf{J}(\mathbf{r}')\cdot\hat{r} - \mathbf{J}(\mathbf{r}')\}}\, e^{+jk_0\hat{r}\cdot\mathbf{r}'}\, dv'$$

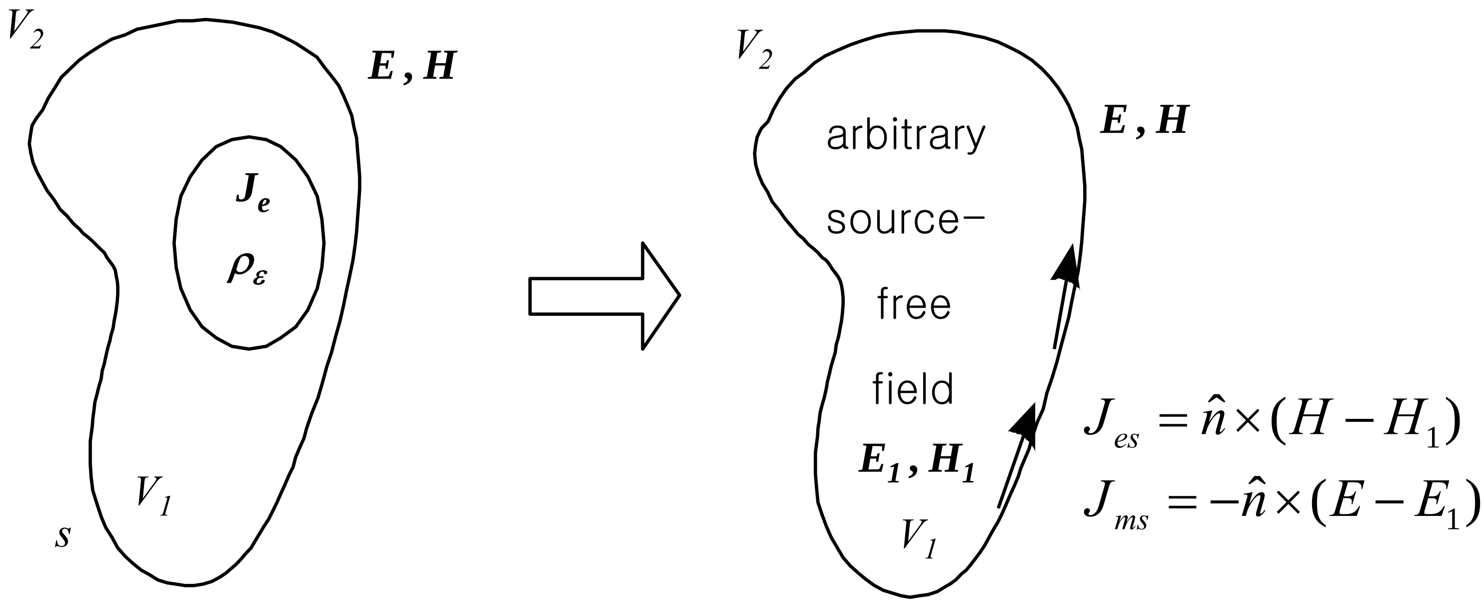
Spherical wave ft

$\frac{e^{-jk_0 r}}{r}$ dependence

Radiation pattern :
angular dependence of radiation

Effective current distribution

Field Equivalence Principles(I)

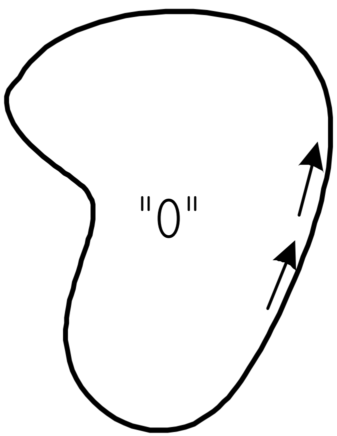


Field Equivalence Principles(II)

case 1 : $E_1 = H_1 = 0$

Huygens' principle

Love's field equivalence principle

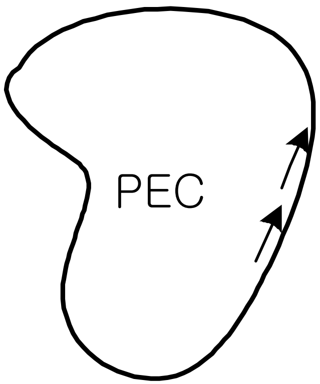


$"G_f"$

$\boldsymbol{J}_{es} = \hat{n} \times \boldsymbol{H}$

$\boldsymbol{J}_{ms} = -\hat{n} \times \boldsymbol{E}$

case 2 : put PEC in V_1

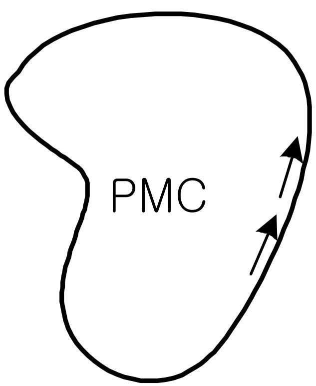


$"G_{PEC}"$

$\boldsymbol{J}_{es}$: short out

$\boldsymbol{J}_{ms} = -\hat{n} \times \boldsymbol{E}$

case 3 : put PMC in V_1



$"G_{PMC}"$

$\boldsymbol{J}_{es} = \hat{n} \times \boldsymbol{H}$

$\boldsymbol{J}_{ms}$: short out

Circular Dish Antenna Pattern(I)

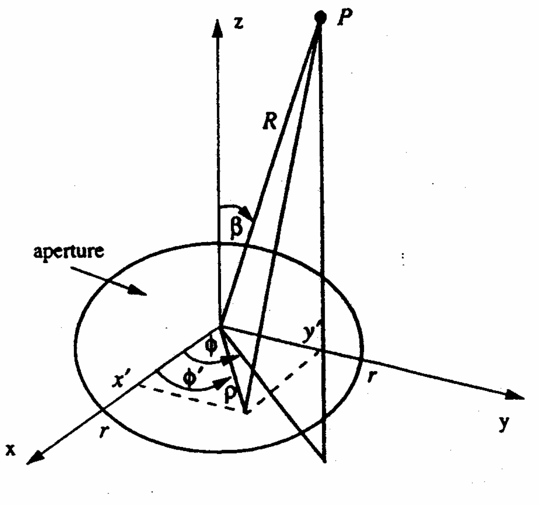


Figure 10.2. Circular aperture geometry.

$$E_a(x',y') = \begin{cases} \hat{x}E_o, & \text{for } |\rho| \; \pi \; r \\ 0, & \text{otherwise} \end{cases}$$

$$H_a(x',y') = \begin{cases} \hat{y}H_o, & \text{for } |\rho| \; \pi \; r \\ 0, & \text{otherwise} \end{cases}$$

equivalent surface currents in the free-space,

$$J_{es}(x',y') = \begin{cases} 2\hat{z}\times\hat{y}H_o = -\hat{x}2E_o, & \text{for } |\rho| \; \pi \; r \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbf{E}(r) = \frac{jk_0 Z_0 e^{-jk_0 r}}{4\pi r} \quad \mathbf{F}(\beta,\phi) = \frac{jk_0 Z_0 e^{-jk_0 r}}{4\pi r} \iint_{aperture} -\hat{x}2E_o e^{+jk_0\hat{r}\cdot\mathbf{r}'} dv'$$

$$F_X(\beta,\phi) = -\iint_{aperture} 2E_o e^{+jk_0\hat{r}\cdot\mathbf{r}'} dx'dy' = -2E_o \iint_{aperture} e^{+jk_0\rho'\sin\beta\cos(\phi-\phi')} \rho' d\rho' d\phi'$$

$$= -2E_o(2\pi)\int_0^r \rho' J_o(k\rho'\sin\beta)d\rho' = (-2E_o)\pi r^2 \frac{2J_1(kr\sin\beta)}{kr\sin\beta}$$

Circular Dish Antenna Pattern(II)

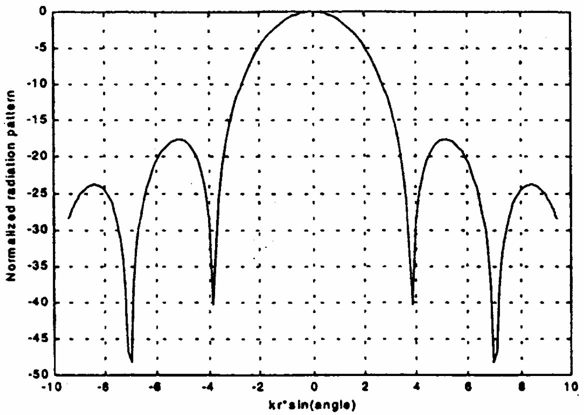


Figure 10.3a. Circular aperture radiation pattern. Typical output produced by "*circ_aperture.m*". $d = 0.3m$; $\lambda = 0.1m$.

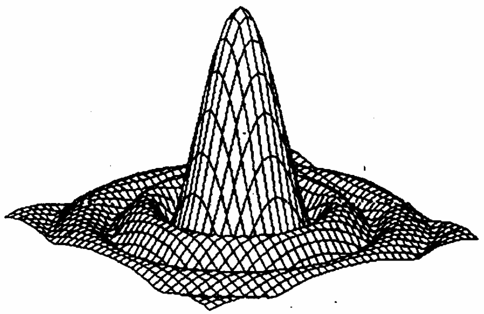


Figure 10.3b. Three-dimensional array pattern corresponding to Fig. 10.3a. Typical output produced by "*circ_aperture.m*". $d = 0.3m$; $\lambda = 0.1m$.

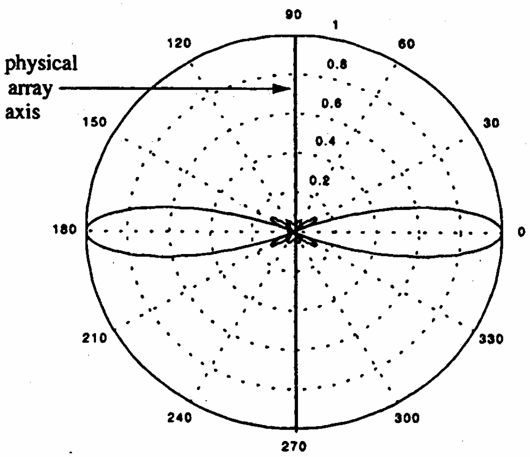
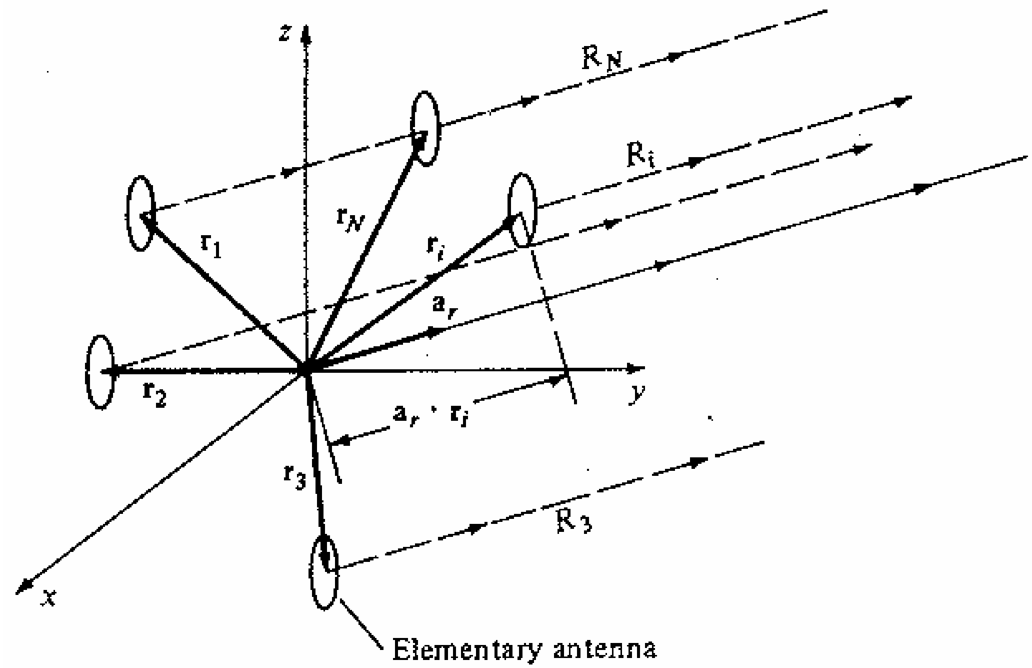


Figure 10.3c. Polar plot for a circular aperture. Typical output produced by "*circ_aperture.m*". $d = 0.3m$; $\lambda = 0.1m$.

Concept of Array Antenna

•Pattern synthesis with wide beam antenna elements

•Use the superposition principle of the field contribution from each current source.



Principle of pattern multiplication

- Field from the reference antenna at the origin with unity excitation :

$$\mathbf{E}(\mathbf{r}) = \frac{e^{-jk_0 r}}{4\pi\, r}\mathbf{F}(\theta,\phi)$$

- Total radiation field : superposition

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} C_i e^{j\alpha_i}\mathbf{F}(\theta,\phi)\frac{e^{-jk_0 R}}{4\pi R}$$

$$= \mathbf{F}(\theta,\phi)\frac{e^{-jk_0 r}}{4\pi\, r}\sum_{i=1}^{N} C_i e^{jk_0 \mathbf{r}'\cdot\hat{r} + j\alpha_i}$$

$$= \frac{e^{-jk_0 r}}{4\pi\, r}\mathbf{F}(\theta,\phi)\times AF(\theta,\phi)$$

$$= \text{element pattern} \times \text{array pattern}$$

Uniform 1-D Array

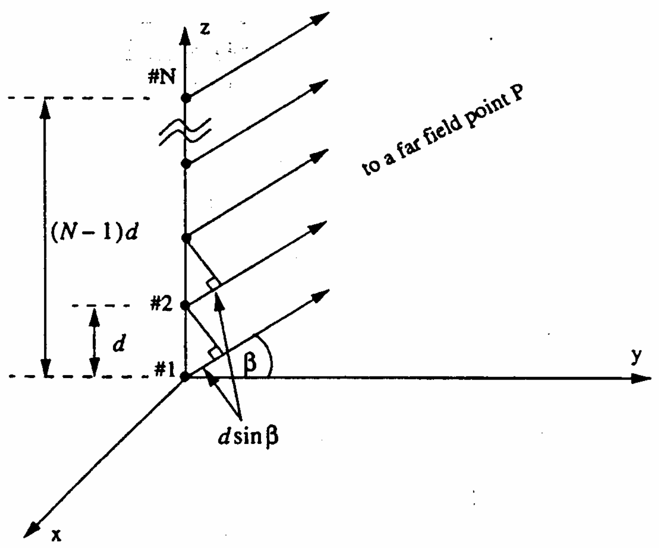


Figure 10.4. Linear array of equally spaced elements.

Uniform 1. constant amplitudes : $C_i = I_0$

2. progressive phase change : $\varphi_i = i\delta$

$$E(\sin\beta) = 1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots + e^{j(n-1)\psi} = \sum_{i=1}^{N} e^{j(i-1)\psi}$$

where $\psi = kd\sin\beta + \delta$

$$\therefore \left|E(\sin\beta)\right| = \left|\frac{\sin(\mathrm{N}(kd\sin\beta+\delta)/2)}{\sin((kd\sin\beta+\delta)/2)}\right|$$

$$\left|E_n(\sin\beta)\right| = \frac{1}{N}\left|\frac{\sin(\mathrm{N}(kd\sin\beta+\delta)/2)}{\sin((kd\sin\beta+\delta)/2)}\right|$$

$$G(\sin\beta) = \left|E_n(\sin\beta)\right|^2 = \frac{1}{N^2}\left(\frac{\sin(\mathrm{N}(kd\sin\beta+\delta)/2)}{\sin((kd\sin\beta+\delta)/2)}\right)^2$$

Broadside array(I)

If the progressive phase, δ=0

$$G(\sin\beta) = \left|E_n(\sin\beta)\right|^2 = \frac{1}{N^2}\left(\frac{\sin(\mathrm{N}(kd\sin\beta)/2)}{\sin((kd\sin\beta)/2)}\right)^2$$

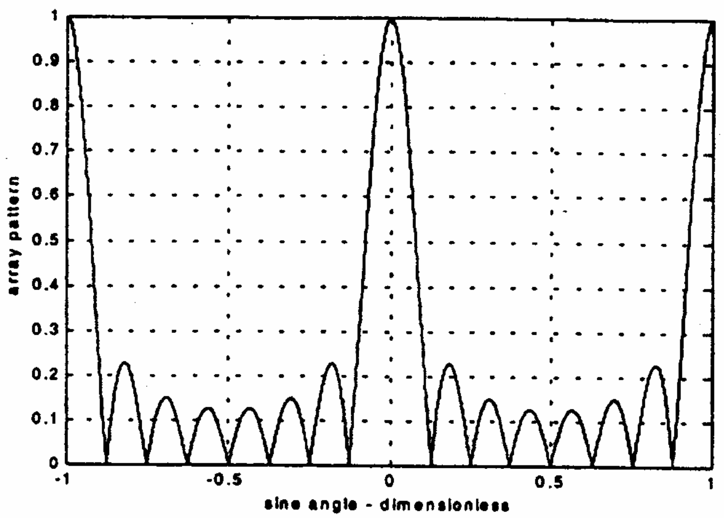


Figure 10.5a. Normalized radiation pattern for a linear array; $N = 8$ and $d = \lambda$.

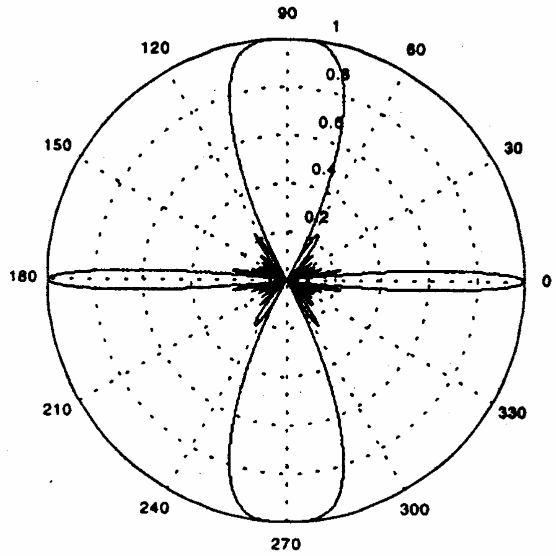


Figure 10.5b. Polar plot for the radiation pattern in Fig. 10.5a.

Broadside array(II)

Beam maxima at

$$kd \sin \beta/2 = \pm m\pi \quad ; m = 0, 1, 2, \cdots$$

$$\beta_{\mathrm{m}} = \mathrm{asin}(\pm \frac{\lambda \mathrm{m}}{\mathrm{d}}) \quad ; m = 0, 1, 2, \cdots$$

$$\begin{cases} \beta_o = 0 \quad ; \text{main beam} \\ \beta_m = \mathrm{asin}(\pm \frac{\lambda \mathrm{m}}{\mathrm{d}}) = \text{real angle for } m \geq 1 \text{ if } d > \lambda \quad ; \text{grating lobes} \end{cases}$$

Beam Steering

Electronical Beam Steering in $\beta=\beta_o$: $\delta=-kd\sin\beta_o$

$$G(\sin\beta)=\left|E_n(\sin\beta)\right|^2=\frac{1}{N^2}\left(\frac{\sin[(\mathrm{N}kd/2)(\sin\beta-\sin\beta_o)]}{\sin[(kd/2)(\sin\beta-\sin\beta_o)]}\right)^2$$

Beam maxima at

$$(kd/2)(\sin\beta-\sin\beta_o)=\pm n\pi \quad ;\mathrm{n}=0,1,2,\ldots$$

$$\mathrm{n}=0\rightarrow \text{main beam at } \beta=\beta_\mathrm{o}$$

$$\mathrm{n}=1,2,3,\ldots\rightarrow \text{grating lobes at } \left|\sin\beta-\sin\beta_o\right|=\pm\frac{\lambda n}{d}$$

$$\text{In order to prevent the grating lobes in } \pm 90^\mathrm{o}, d\ \pi\ \frac{\lambda}{2}$$

Side lobes and nulls(Broadside array)

– Side lobes at

$$\frac{\mathrm{N}kd\sin\beta}{2} = \pm(2l+1)\frac{\pi}{2} \quad ;l = 1, 2, \ldots$$

$$\beta_l = \mathrm{asin}(\pm\frac{\lambda}{2d}\frac{2l+1}{N}) \quad ;l = 1, 2, \ldots$$

– Nulls at

$$\frac{\mathrm{N}kd\sin\beta}{2} = \pm n\pi \quad ;n = 1, 2, \ldots, n \neq \mathrm{N}, 2\mathrm{N}, \ldots$$

$$\beta_n = \mathrm{asin}(\pm\frac{\lambda}{d}\frac{n}{N}) \quad ;n = 1, 2, \ldots, n \neq \mathrm{N}, 2\mathrm{N}, \ldots$$

– Half power point at

$$\frac{\mathrm{N}kd\sin\beta}{2} = 1.391 rad \rightarrow \beta_h = \mathrm{asin}(\frac{\lambda}{2\pi d}\frac{2.782}{N})$$

Array Patterns w/o grating lobe

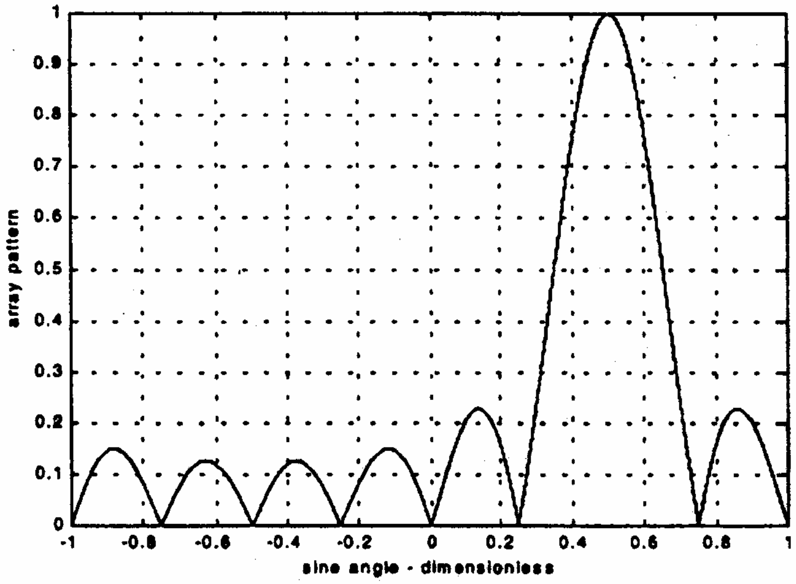


Figure 10.6a. Normalized radiation pattern for a linear array. $N = 8$, $d = \lambda/2$, and $\beta_0 = 30°$.

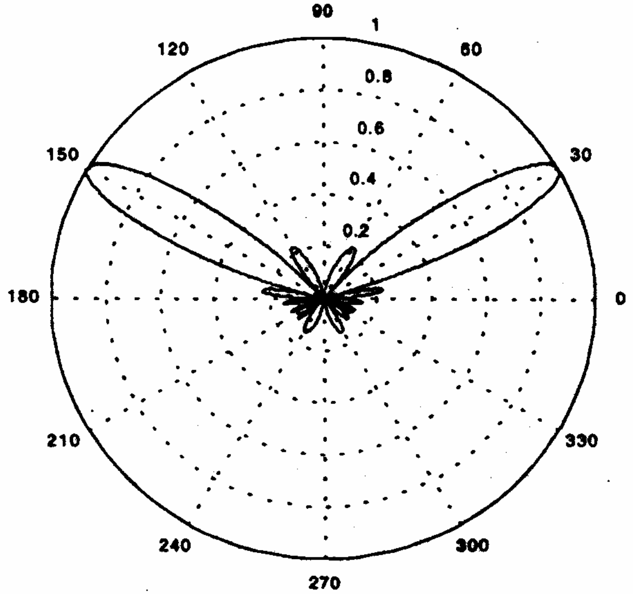


Figure 10.6b. Polar plot corresponding to Fig. 10.6a.

Array Patterns w grating lobes

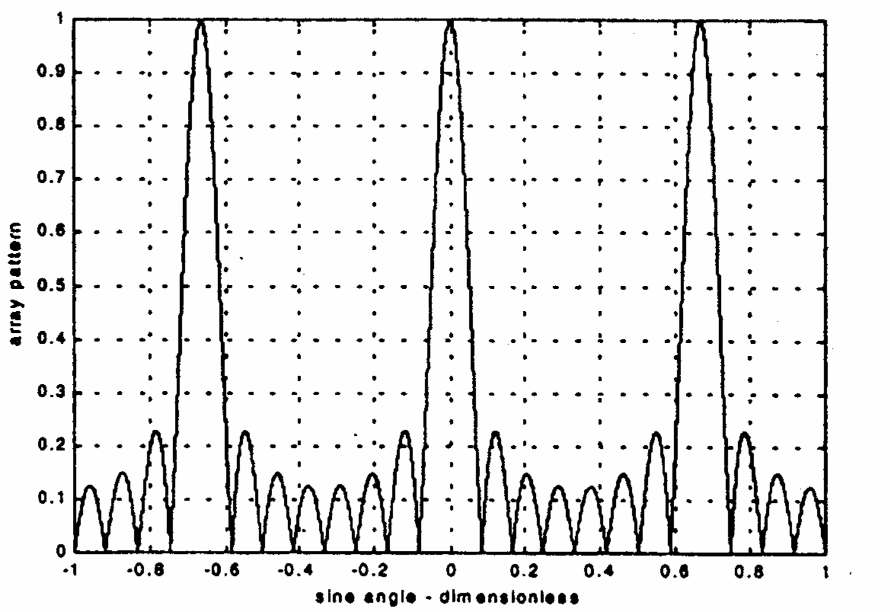


Figure 10.7a. Normalized radiation pattern for a linear array. $N = 8$, $d = 1.5\lambda$, **and** $\beta_0 = 0°$.

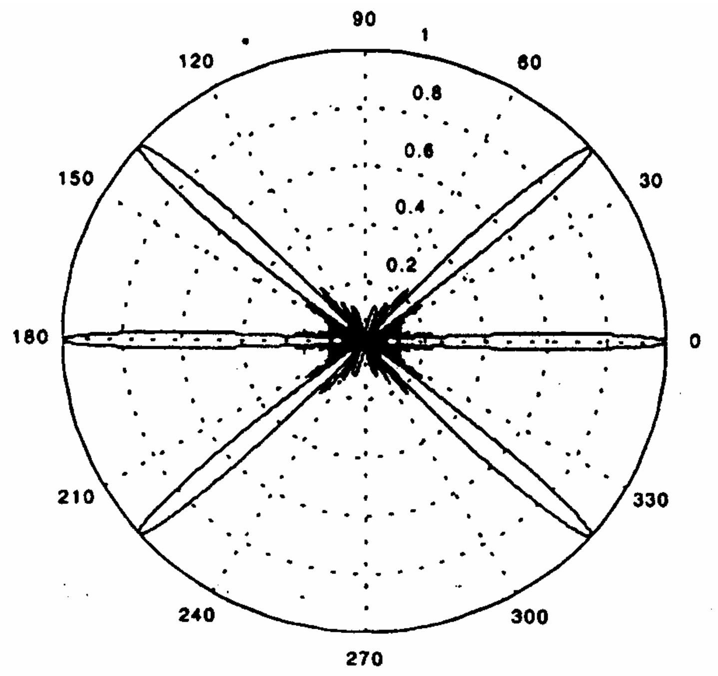


Figure 10.7b. Polar plot corresponding to Fig. 10.7a.

Array Tapering

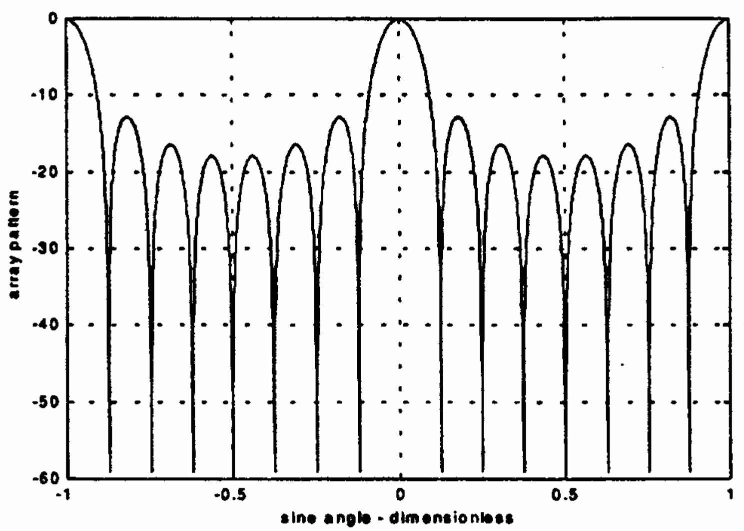


Figure 10.8. Normalized radiation pattern for a linear array. $N = 8$ and $d = \lambda/2$.

$\text{SLL} = 13.46\ \text{dB}$

$\rightarrow$ Fourier Transform of uniform current distribution

To reduce the SLL, taper the current distribution over the face of array

$\rightarrow$ widen the main beam

Computation of the Radiation Pattern via the DFT

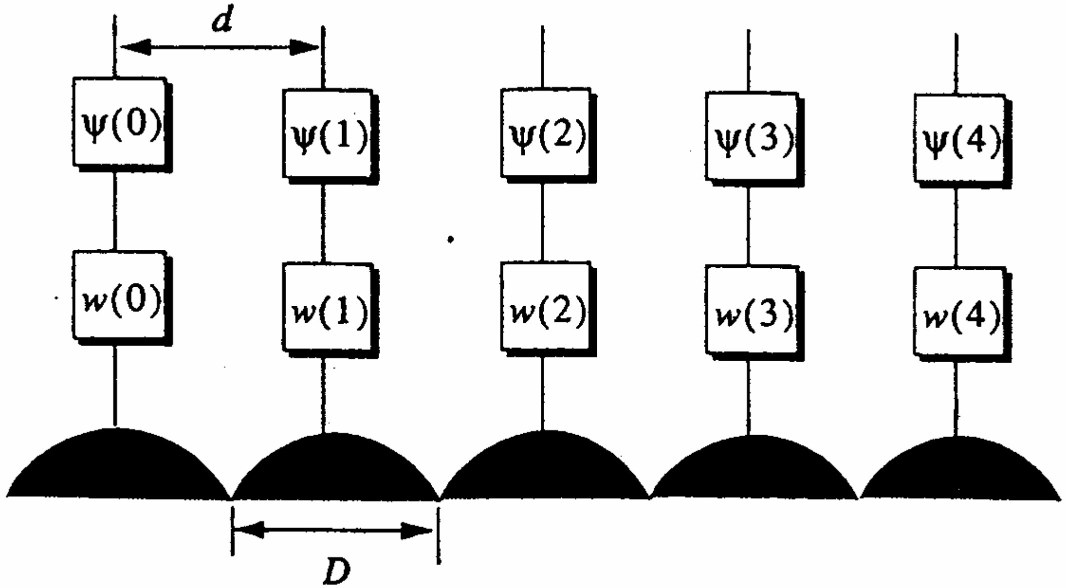


Figure 10.9. Linear array of size 5, with tapering and phase shifting hardware.

Circular dish array of diameter D=d
element spacing d
w(n) : tapering sequence
Ψ(n) : phase shifting sequence

Array Pattern

When the phase reference is taken as the physical center of the array

$$E(\sin\beta) = \sum_{n=0}^{N-1} w(n) e^{j\Delta\phi(n-(\frac{N-1}{2}))} = e^{j\phi_0} \sum_{n=0}^{N-1} w(n) V_1^n ------(1)$$

where $\Delta\phi = \frac{2\pi d}{\lambda}\sin\beta,\ \phi_0 = \frac{(N-1)\Delta\phi}{2},\ V_1^n = e^{-jn\Delta\phi}$

DFT of the sequence *w(n)*

$$W(k) = \sum_{n=0}^{N-1} w(n) e^{-j\frac{2\pi nk}{N}}, k = 0,1,2,....,(N-1) ---------(2)$$

Comparing (1) with (2), $E(\sin\beta) = e^{j\phi_0} W(k)$

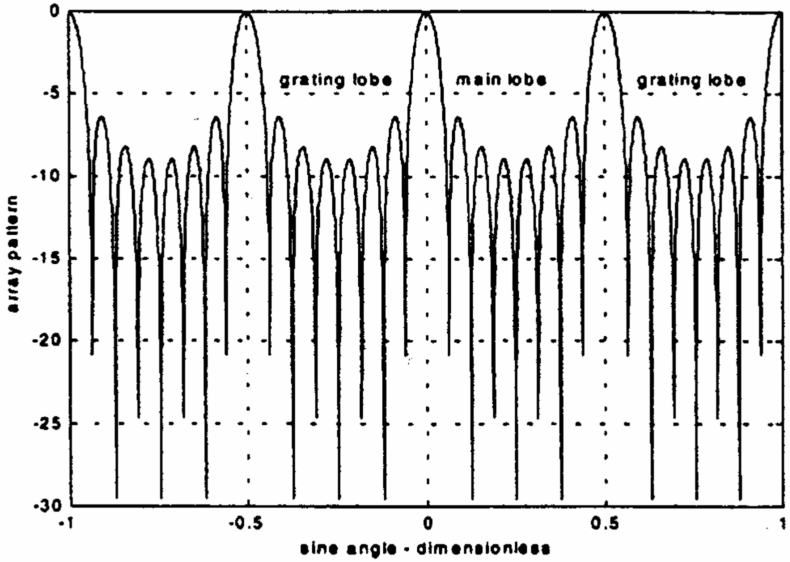
where $\sin\beta_k = \frac{\lambda k}{Nd}, k = 0,1,2,...,(N-1)$

$\rightarrow$ array factor = DFT of $w(n)$

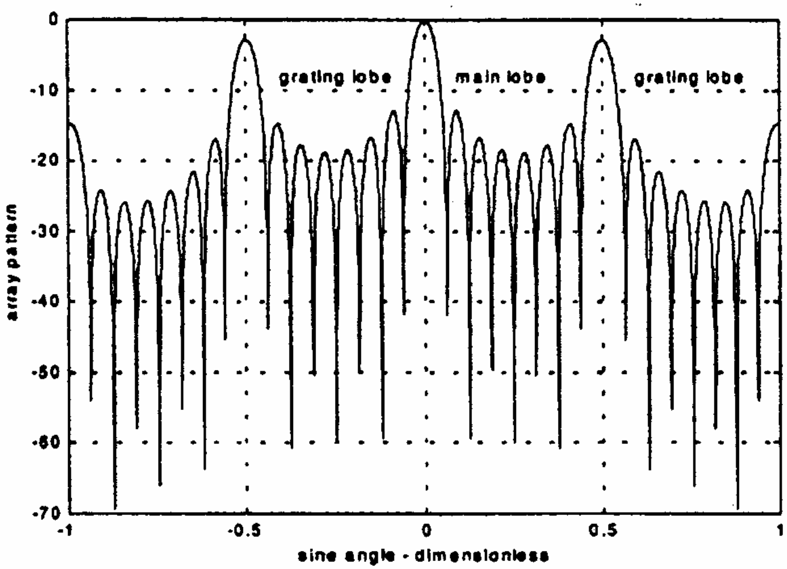
$\therefore$ one-way radiation pattern

$$G(\sin\beta) = G_e(\frac{\lambda k}{Nd})\left|W(k)\right|$$

Normalized one-way pattern



isotropic elements



circular dishes

Figure 10.10. Normalized one-way pattern for linear array of size 8, isotropic elements, and circular dishes. This plot can be reproduced using MATLAB program "fig10_10.m" given in Listing 10.4 in Section 10.9.

Rectangular Planar Array

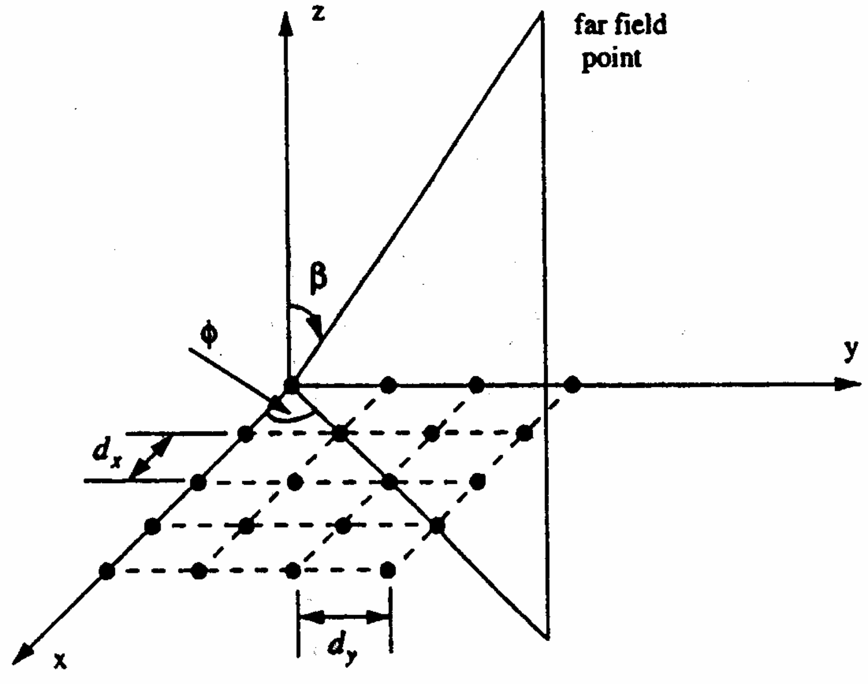


Figure 10.11. Planar array geometry.

Array Pattern

Excitation of 2 - D arrays :

General 2 - D Array : $C_{nm} e^{j\varphi_{xn} + j\varphi_{ym}}$

Uniform 2 - D Array : $C_{nm} = 1, \varphi_{xn} = n\alpha d_x, \varphi_{ym} = m\beta d_y \rightarrow e^{jn\alpha d_x + jn\beta d_y}$

Broadside 2 - D Array : $C_{nm} = 1, \varphi_{xn} = \varphi_{ym} = 0 \rightarrow 1$

$$E(\beta,\phi) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} E_e(\theta,\phi)\frac{e^{-jk(r-\hat{r}\cdot r'_{nm})}}{4\pi r}$$

$$= E_e(\theta,\phi)\frac{e^{-jkr}}{4\pi r}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{j(knd_x \sin\beta\cos\phi + kmd_y \sin\beta\sin\phi)}$$

$$= E_e(\theta,\phi)\frac{e^{-jkr}}{4\pi r}\sum_{n=0}^{N-1} e^{jknd_x \sin\beta\cos\phi}\sum_{m=0}^{N-1} e^{jkmd_y \sin\beta\sin\phi}$$

$$= E_e(\theta,\phi)\frac{e^{-jkr}}{4\pi r}\left|\frac{\sin((Nkd_x \sin\beta\cos\phi)/2)}{\sin(kd_x \sin\beta\cos\phi/2)}\right|\left|\frac{\sin((Nkd_y \sin\beta\sin\phi)/2)}{\sin(kd_y \sin\beta\sin\phi/2)}\right|$$

$= \text{Element Pattern} \times \text{Array Factor}$

Radiation Pattern

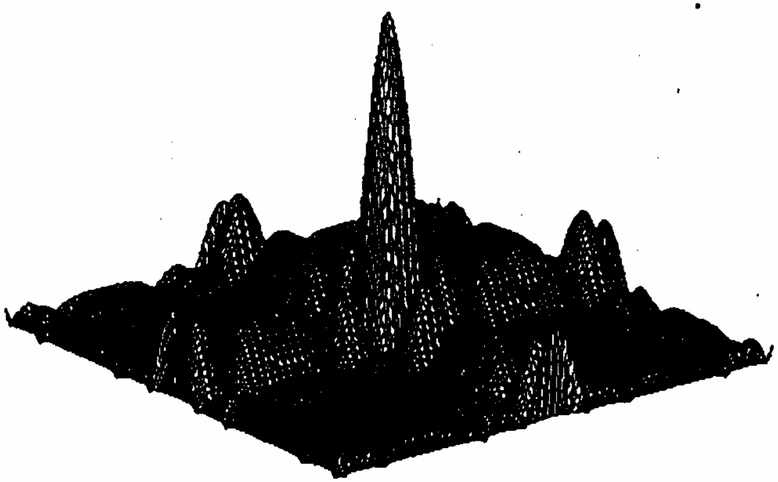


Figure 10.12a. Three-dimensional pattern for a rectangular array of size 5x5, and uniform element spacing.

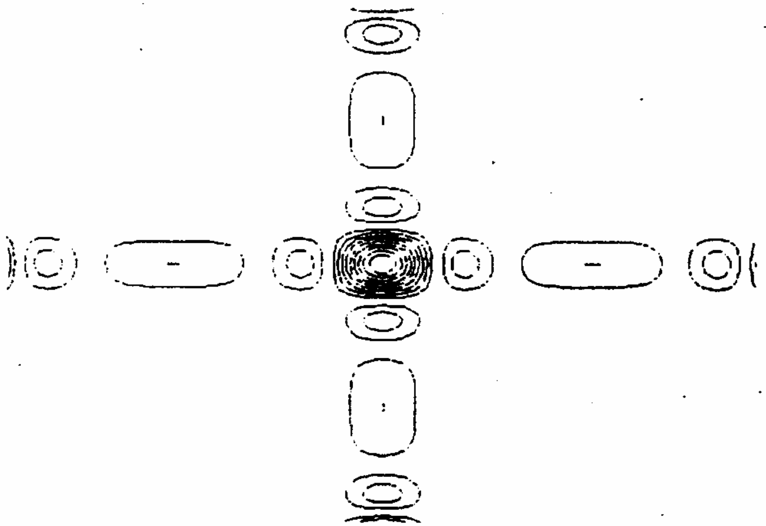


Figure 10.12b. Contour plot corresponding to Fig. 10.12a.

5x5 array of isotropic elements with spacing $d_x = d_y = \lambda/2$

Adaptive Array

Adaptive array:
- eliminate unwanted signals
- enhance desired target return
  by calculating complex weights and applying to each channel
- can be implemented in RF, IF, BB, or digital level

Successful implementation depends on
- a proper choice of the reference signal used for comparison against the received target/jammer returns
- fast (real time) computation of the optimum weights : time required for complex matrix inversion

Conventional Beamforming

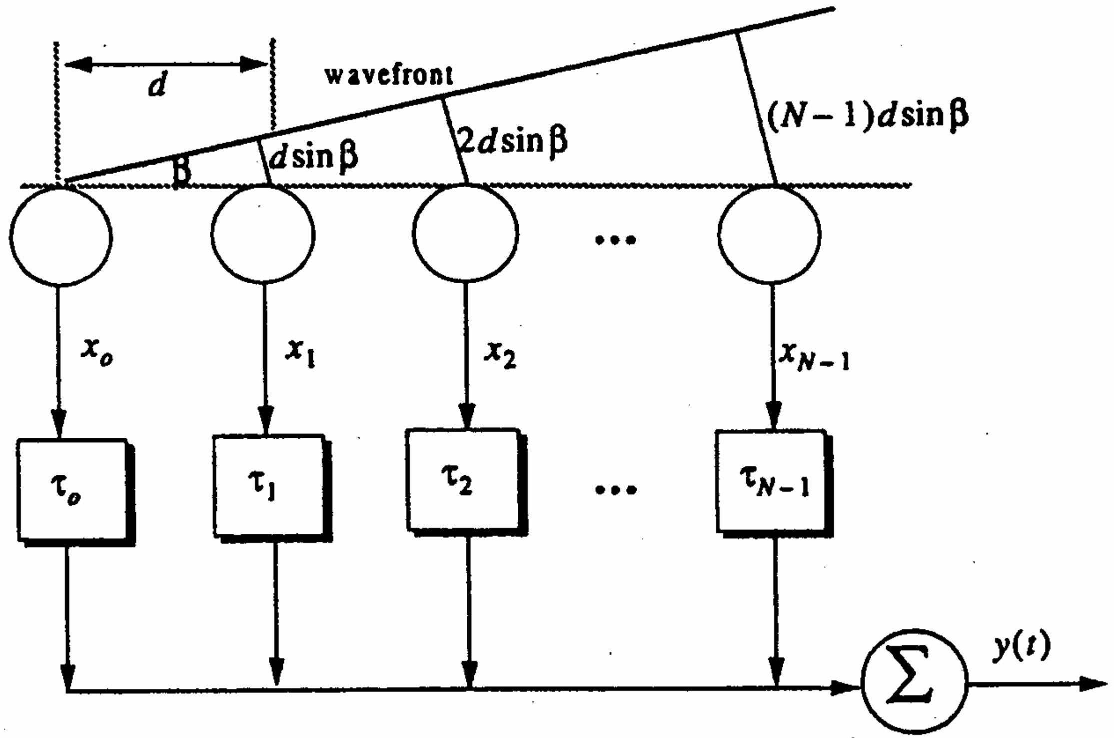


Figure 10.13. A linear array of size N, element spacing d, and an incident plane wave defined by $\sin\beta$.

Conventional Beamformer :single input

$$y(t) = \sum_{n=0}^{N-1} x_n(t-\tau_n),$$

$$\tau_n = (N-1-n)\frac{d}{c}\sin\beta; \quad n = 0,1,\ldots N-1$$

where d is the element spacing and c is the speed of light.

$$\vec{Y}(\omega) = \sum_{n=0}^{N-1} X_n(\omega)e^{-j\omega\tau_n} \quad = \vec{a}^{+}\vec{X}$$

$$\vec{a}^{+} = [e^{-j\omega\tau_0}, e^{-j\omega\tau_1}, \ldots e^{-j\omega\tau_{N-1}}] --------------- (1)$$

$$\vec{X}^{+} = [X_0(\omega), X_1(\omega), \ldots X_{N-1}(\omega)]^{*}$$

If A_1 is the amplitude of the wavefront defined by $\sin\beta_1$

$$\vec{X} = A_1\vec{s}_{k1}^{*}$$

where $\vec{s}_k$ is a steering vector given by

$$\vec{s}_{k1}^{+} = [1, e^{-jk}, e^{-j2k}, \ldots, e^{-j(N-1)k}]; k = \frac{2\pi d}{\lambda}\sin\beta --- (2)$$

Comparing (1) and (2), $\vec{a} = e^{j(N-1)k}\vec{s}_k \xrightarrow[\text{neglect phase}]{} \vec{s}_k$

The beam former output is

$$\vec{Y} = \vec{a}^{+}\vec{X} = \vec{s}_k^{+}A_1\vec{s}_{k1}^{*} = A_1\vec{s}_k^{+}\vec{s}_{k1}^{*}$$

The array pattern of the beam steered at k_1 is

$$S(k) = E[\vec{Y}\vec{Y}^{+}] = E[A_1\vec{s}_k^{+}\vec{s}_{k1}^{*}\vec{s}_{k1}^{t}\vec{s}_k A_1^{*}] = P_1\vec{s}_k^{+}\Re\vec{s}_k$$

where $P_1 = E[|A_1|^2]$ and $\Re$ is the correlation matrix, $\vec{s}_{k1}^{*}\vec{s}_{k1}^{t}$.

Conventional Beamformer:multiple inputs

Consider L incident plane waves with directions of arrival defined by

$$k_i = \frac{2\pi d}{\lambda} \sin \beta_i ; i = 1, L$$

The n^th^ sample at the output of the m^th^ sensor is

$$y_m(n) = v(n) + \sum_{i=1}^{L} A_i(n) e^{-jmk_i} \;; m = 0, N-1$$

where $A_i(n)$ is the amplitude of the i^th^ plane wave, and $v(n)$ is noise.

$$\vec{y}(n) = \vec{v}(n) + \sum_{i=1}^{L} A_i(n) \vec{s}_{ki}^{\,*} = \vec{v}(n) + \aleph^* \vec{A}(n)$$

steering matrix $\aleph = [\vec{s}_{k1}, \vec{s}_{k2}, \ldots, \vec{s}_{kL}]$; $N \times L$ matrix

Autocorrelation matrix of the field measured by the array is

$$\Re = E\{\vec{y}(n)\vec{y}(n)^+\} = E\{[\vec{v}(n) + \aleph^* \vec{A}(n)][\vec{v}(n)^+ + \vec{A}(n)^+ \aleph^t]\}$$

$$= E\{\vec{v}(n)\vec{v}(n)^+\} + E\{\aleph^* \vec{A}(n) \vec{A}(n)^+ \aleph^t\} = \sigma_v^2 I + \aleph^* C \aleph^t$$

where $C = \mathrm{dig}[P_1, P_2, \ldots, P_L]$.

The array pattern can be found by standard spectral estimators.

$\hat{S}(k) = \vec{s}_k^{\,+} \Re \vec{s}_k$ (Barlett beamformer)

It has spectral peaks at β_i for each wavefront defined by k_i.

The SNR for i^th^ wavefront is given by $SNR = N(\frac{P_i}{\sigma_v^2})$